Functions of several variables

(M.Sc. (MATHEMATICS), Paper - VI)

(Real Analysis - II)

Lecture - 02

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Functions of two variables

Limits and continuity :→

Let S is a subset of $\mathbb{R}^n$. consider a function $f : S \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$, and if $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ and $b = (b_1, \ldots, b_m) \in \mathbb{R}^m$

Then

$$\lim_{x \to a} f(x) = b \quad (\text{or } f(x) \to b \text{ as } x \to a) \quad -①$$

to mean that

$$\lim_{\|x-a\| \to 0} \|f(x) - b\| = 0. \quad — ②$$

(It is not required that f be defined at the point a)

If we write $h = x - a$. Equation ② becomes

$$\lim_{\|h\| \to 0} \|f(x+h) - b\| = 0$$

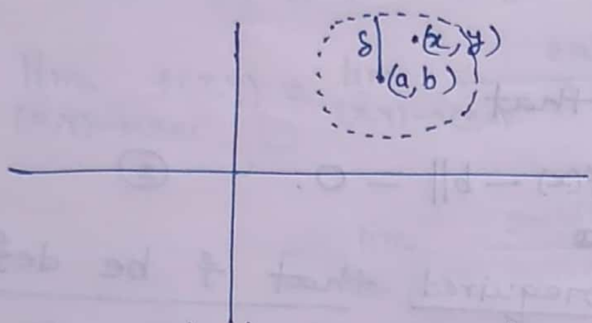
→ A function f is said to be continuous at 'a' if f is defined at a and if

$$\lim_{x \to a} f(x) = f(b)$$

We say f is continuous on a set S if f is continuous at each point of S.

Definition :→ A function $f : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ is said to tend to limit l as $(x,y) \to (a,b)$ if for every arbitrary $\epsilon > 0$, $\exists$ a positive number $\delta(\epsilon)$, such that

$$|f(x,y) - l| < \epsilon, \text{ whenever } 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$$



symbolically we write

$$\lim_{(x,y) \to (a,b)} f(x,y) = l$$

where l is called the double limit or simultaneous limit of f when $(x,y) \longrightarrow (a,b)$.

Remark :→ If $\lim_{(x,y) \to (a,b)} f(x,y) = l$ and $y = \phi(x)$ is any function such that $\phi(x) \longrightarrow b$, when $x \to a$, then $f(x, \phi(x))$, must exists & be equal to l.

Thus if we can find two functions $\phi_1(x)$ and $\phi_2(x)$ such that the limit $f(x, \phi_1(x))$ and $f(x, \phi_2(x))$ are different, then the simultaneous limit does not exists.

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Example: → Let $f(x,y) = \begin{cases} \dfrac{x^2 y}{x^4+y^2}, & \text{if } x^4+y^2 \neq 0 \\ 0, & \text{if } x+y=0 \end{cases}$

Then

$(x,y) \to (0,0)$ along x-axis i.e; $y=0$ & $x \to 0$

$$\therefore \lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4+y^2}$$

$$= \lim_{x\to 0} \frac{0}{x^4} = 0$$

Similarly $(x,y) \to (0,0)$ along y-axis, i.e; $x=0, y\to 0$

$$\therefore \lim_{(x,y)\to(0,0)} f(x,y) = \lim_{y\to 0} \frac{0}{y^2} = 0$$

When $(x,y) \longrightarrow (0,0)$ along any line $y = mx$, then

$$f(x,y) = f(x, mx) = \frac{mx^3}{x^4 + m^2x^2} = \frac{mx}{x^2+m^2} \to 0$$

as $x \to 0$

So, any straight line approach gives,

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0$$

When $(x,y) \to (0,0)$ along the path $y = mx^2$.

$$\therefore \lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} f(x, mx^2)$$

$$= \lim_{x\to 0} \frac{mx^4}{x^4 + m^2x^4}$$

$$= \lim_{x\to 0} \frac{m}{1+m^2} = \frac{m}{1+m^2}$$

Which is different for different m.

Hence $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exists.

Example ③:→ Show that

$$\lim_{(x,y)\to(0,0)} \frac{2xy^2}{x^2+y^4} \text{ does not exists.}$$

Solution:→ When $(x,y)\to(0,0)$ along the path $x = my^2$ and $y\to 0$, we get

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{2xy^2}{x^2+y^4}$$

$$= \lim_{y\to 0} \frac{2my^4}{(m^2+1)y^4}$$

$$= \frac{2m}{1+m^2}$$

Which is different for different m.

Hence $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exists.